CHAPTER 13
RATIO, PROPORTION, AND VARIATION

The solution of problems based on ratio, proportion, and variation involves no new principles. However, familiarity with these topics will often lead to quick and simple solutions to problems that would otherwise be more complicated.

RATIO

The results of observation or measurement often must be compared with some standard value in order to have any meaning. For example, to say that a man can read 400 words per minute has little meaning as it stands. However, when his rate is compared to the 250 words per minute of the average reader, one can see that he reads considerably faster than the average reader. How much faster? To find out, his rate is divided by the average rate, as follows:

$$\frac{400}{250} = \frac{8}{5}$$

Thus, for every 5 words read by the average reader, this man reads 8. Another way of making this comparison is to say that he reads $1\frac{3}{5}$ times as fast as the average reader.

When the relationship between two numbers is shown in this way, they are compared as a RATIO. A ratio is a comparison of two like quantities. It is the quotient obtained by dividing the first number of a comparison by the second.

Comparisons may be stated in more than one way. For example, if one gear has 40 teeth and another has 10, one way of stating the comparison would be 40 teeth to 10 teeth. This comparison could be shown as a ratio in four ways as follows:

1. 40:10
2. $40 \div 10$
3. $\frac{40}{10}$
4. The ratio of 40 to 10.

When the emphasis is on "ratio," all of these expressions would be read, "the ratio of 40 to 10." The form $40 \div 10$ may also be read "40 divided by 10." The form $\frac{40}{10}$ may also be read "40 over 10."

Comparison by means of a ratio is limited to quantities of the same kind. For example, in order to express the ratio between 6 ft and 3 yd, both quantities must be written in terms of the same unit. Thus the proper form of this ratio is 2 yd : 3 yd, not 6 ft : 3 yd. When the parts of the ratio are expressed in terms of the same unit, the units cancel each other and the ratio consists simply of two numbers. In this example, the final form of the ratio is 2 : 3.

Since a ratio is also a fraction, all the rules that govern fractions may be used in working with ratios. Thus, the terms may be reduced, increased, simplified, and so forth, according to the rules for fractions. To reduce the ratio 15:20 to lowest terms, write the ratio as a fraction and then proceed as for fractions. Thus, 15:20 becomes

$$\frac{15}{20} = \frac{3}{4}$$

Hence the ratio of 15 to 20 is the same as the ratio of 3 to 4.

Notice the distinction in thought between $\frac{3}{4}$ as a fraction and $\frac{3}{4}$ as a ratio. As a fraction we think of $\frac{3}{4}$ as the single quantity "three-fourths." As a ratio, we think of $\frac{3}{4}$ as a comparison between the two numbers, 3 and 4. For example, the lengths of two sides of a triangle are $1\frac{9}{16}$ ft and 2 ft. To compare these lengths by means of a ratio, divide one number by the other and reduce to lowest terms, as follows:

$$\frac{1\frac{9}{16}}{2} = \frac{\frac{25}{16}}{2} = \frac{25}{32}$$

The two sides of the triangle compare as 25 to 32.

INVERSE RATIO

It is often desirable to compare the numbers of a ratio in the inverse order. To do this, we simply interchange the numerator and the denominator. Thus, the inverse of 15:20 is 20:15. When the terms of a ratio are interchanged, the INVERSE RATIO results.

Practice problems. In problems 1 through 6, write the ratio as a fraction and reduce to lowest terms. In problems 7 through 10, write the inverse of the given ratio.

1. The ratio of 5 lb to 15 lb
2. \$16 : \$12
3. $16 \div 4$
4. One quart to one gallon
5. 5x to 10x
6. $3\frac{1}{3} : 4\frac{1}{2}$
7. The ratio of 6 ft to 18 ft
8. $\frac{4}{8}$
9. 5 : 8
10. 15 to 21

Answers:

1. $\frac{1}{3}$	6. $\frac{20}{27}$
2. $\frac{4}{3}$	7. $\frac{3}{1}$
3. $\frac{4}{1}$	8. $\frac{2}{1}$
4. $\frac{1}{4}$	9. $\frac{8}{5}$
5. $\frac{1}{2}$	10. $\frac{7}{5}$

PROPORTION

Closely allied with the study of ratio is the subject of proportion. A PROPORTION is nothing more than an equation in which the members are ratios. In other words when two ratios are set equal to each other, a proportion is formed. The proportion may be written in three different ways as in the following examples:

$$15:20 :: 3:4$$

$$15:20 = 3:4$$

$$\frac{15}{20} = \frac{3}{4}$$

The last two forms are the most common. All these forms are read, "15 is to 20 as 3 is to 4." In other words, 15 has the same ratio to 20 as 3 has to 4.

One reason for the extreme importance of proportions is that if any three of the terms are given, the fourth may be found by solving a simple equation. In science many chemical and physical relations are expressed as proportions. Consequently, a familiarity with proportions will provide one method for solving many applied problems. It is evident from the last form shown, $\frac{15}{20} = \frac{3}{4}$, that a proportion is really a fractional equation. Therefore, all the rules for fraction equations apply.

TERMS OF A PROPORTION

Certain names have been given to the terms of the two ratios that make up a proportion. In a proportion such as 3:8 = 9:24, the first and the last terms (the outside terms) are called the EXTREMES. In other words, the numerator of the first ratio and the denominator of the second are called the extremes. The second and third terms (the inside terms) are called the MEANS. The means are the denominator of the first ratio and the numerator of the second. In the example just given, the extremes are 3 and 24; the means are 8 and 9.

Four numbers, such as 5, 8, 15, and 24, form a proportion if the ratio of the first two in the order named equals the ratio of the second two. When these numbers are set up as ratios with the equality sign between them, the members will reduce to an identity if a true proportion exists. For example, consider the following proportion:

$$\frac{5}{8} = \frac{15}{24}$$

In this proportion, $\frac{15}{24}$ must reduce to $\frac{5}{8}$ for the proportion to be true. Removing the same factor from both members of $\frac{15}{24}$ we have

$$\frac{5}{8} = \frac{3(5)}{3(8)}$$

The number 3 is the common factor that must be removed from both the numerator and the denominator of one fraction in order to show that the expression

$$\frac{5}{8} = \frac{15}{24}$$

is a true proportion. To say this another way, it is the factor by which both terms of the ratio $\frac{5}{8}$ must be multiplied in order to show that this ratio is the same as $\frac{15}{24}$.

Practice problems. For each of the following proportions, write the means, the extremes, and the factor of proportionality.

1. $\frac{3}{16} = \frac{15}{80}$
2. 4:5 = 12:15
3. $\frac{25}{75} = \frac{1}{3}$
4. 12:3 :: 4:1

Answers:

1. Means: 16 and 15
   Extremes: 3 and 80
   Factor of proportionality: 5
2. M: 5 and 12
   E: 4 and 15
   FP: 3
3. M: 75 and 1
   E: 25 and 3
   FP: 25
4. M: 3 and 4
   E: 12 and 1
   FP: 3

OPERATIONS OF PROPORTIONS

It is often advantageous to change the form of a proportion. There are rules for changing or combining the terms of a proportion without altering the equality between the members. These rules are simplifications of fundamental rules for equations; they are not new, but are simply adaptations of laws or equations presented earlier in this course.

Rule 1. In any proportion, the product of the means equals the product of the extremes.

This is perhaps the most commonly used rule of proportions. It provides a simple way to rearrange a proportion so that no fractions are present. In algebraic language the rule is illustrated as follows:

$$\frac{a}{b} = \frac{c}{d}$$

$$bc = ad$$

To prove this rule, we note that the LCD of the two ratios $\frac{a}{b}$ and $\frac{c}{d}$ is bd. Multiplying both members of the equation in its original form by this LCD, we have

$$bd \cdot \frac{a}{b} = bd \cdot \frac{c}{d}$$

$$ad = bc$$

The following numerical example illustrates the simplicity of rule 1:

$$\frac{3}{8} = \frac{9}{24}$$

$$8(9) = 3(24)$$

If one of the terms of a proportion is a variable to the first power as in

$$7:5 = x:6$$

the proportion is really a linear equation in one variable. Such an equation can be solved for the unknown.

Equating the products of the means and extremes produces the following:

$$5x = 42$$

$$x = 8\tfrac{2}{5}$$

Mean Proportional

When the two means of a proportion are the same quantity, that quantity is called the MEAN

PROPORTIONAL between the other two terms. In the proportion

$$\frac{a}{x} = \frac{x}{c}$$

x is the mean proportional between a and c.

Rule 2. The mean proportional between two quantities is the square root of their product. This rule is stated algebraically as follows:

$$\frac{a}{x} = \frac{x}{c}$$

$$x = \pm\sqrt{ac}$$

To prove rule 2, we restate the proportion and apply rule 1, as follows:

$$\frac{a}{x} = \frac{x}{c}$$

$$x^2 = ac$$

$$x = \pm\sqrt{ac}$$

Rule 2 is illustrated by the following numerical example:

$$\frac{2}{8} = \frac{8}{32}$$

$$8 = \sqrt{2(32)}$$

$$8 = \sqrt{64}$$

OTHER FORMS FOR PROPORTIONS

If four numbers, for example, a, b, c, and d, form a proportion, such as

$$\frac{a}{b} = \frac{c}{d}$$

they also form a proportion according to other arrangements.

Inversion

The four selected numbers are in proportion by INVERSION in the form

$$\frac{b}{a} = \frac{d}{c}$$

The inversion relationship is proved as follows, by first multiplying both members of the original proportion by $\frac{bd}{ac}$:

$$\left(\frac{bd}{ac}\right)\left(\frac{a}{b}\right) = \left(\frac{bd}{ac}\right)\left(\frac{c}{d}\right)$$

$$\frac{d}{c} = \frac{b}{a}$$

Note that the product of the means and the product of the extremes still yield the same equality as in the original proportion.

The inversion relationship may be illustrated by the following numerical example:

$$\frac{5}{8} = \frac{10}{16}$$

Therefore,

$$\frac{8}{5} = \frac{16}{10}$$

Alternation

The four selected numbers (a, b, c, and d) are in proportion by ALTERNATION in the following form:

$$\frac{a}{c} = \frac{b}{d}$$

To prove the alternation relationship, first multiply both sides of the original proportion by $\frac{b}{c}$, as follows:

$$\frac{a}{b} = \frac{c}{d}$$

$$\frac{b}{c}\left(\frac{a}{b}\right) = \frac{b}{c}\left(\frac{c}{d}\right)$$

$$\frac{a}{c} = \frac{b}{d}$$

The following numerical example illustrates alternation:

$$\frac{5}{8} = \frac{10}{16}$$

Therefore,

$$\frac{5}{10} = \frac{8}{16}$$

SOLVING PROBLEMS BY MEANS OF PROPORTION

One of the most common types of problems based on proportions involves triangles with proportional sides. Suppose that the corresponding sides of two triangles are known to be proportional. (See fig. 13-1.) The lengths of the sides of one triangle are 8, 9, and 11. The length of the side of the second triangle corresponding to side 8 in the first triangle is 10. We wish to find the lengths of the remaining sides, b and c.

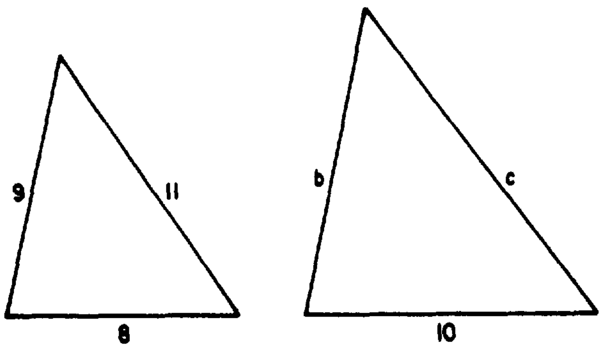

Figure 13-1.—Triangles with corresponding sides proportional.

Since the corresponding sides are proportional, the pairs of corresponding sides may be used to form proportions as follows:

$$\frac{8}{10} = \frac{9}{b}$$

$$\frac{9}{b} = \frac{11}{c}$$

$$\frac{8}{10} = \frac{11}{c}$$

To solve for b, we use the proportion

$$\frac{8}{10} = \frac{9}{b}$$

and obtain the following result:

$$8b = 90$$

$$4b = 45$$

$$b = 11\frac{1}{4}$$

The solution for c is similar to that for b, using the proportion

$$\frac{8}{10} = \frac{11}{c}$$

with the following result:

$$8c = 110$$

$$c = 13\frac{3}{4}$$

The sides of the second triangle are 10, $11\frac{1}{4}$, and $13\frac{3}{4}$. The result can also be obtained by using the factor of proportionality. Since 8 and 10 are lengths of corresponding sides, we can write

$$8k = 10$$

$$k = \frac{10}{8} = \frac{5}{4}$$

The factor of proportionality is thus found to be $\frac{5}{4}$.

Multiplying any side of the first triangle by $\frac{5}{4}$ gives the corresponding side of the second triangle, as follows:

$$b = 9\left(\frac{5}{4}\right) = \frac{45}{4} = 11\frac{1}{4}$$

$$c = 11\left(\frac{5}{4}\right) = \frac{55}{4} = 13\frac{3}{4}$$

Proportional sides of similar triangles may be used to determine the height of an object by measuring its shadow. (See fig. 13-2.)

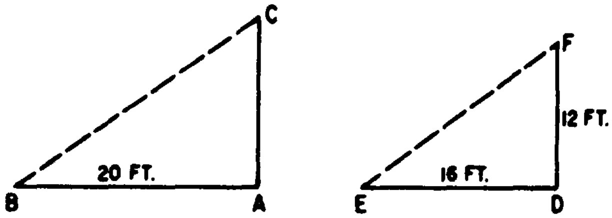

Figure 13-2.—Measuring height by shadow length.

In figure 13-2, mast AC casts a shadow 20 ft. long (AB). At the same time, DF (12 ft. long) casts a shadow of 16 ft. long (DE). Assuming that both masts are vertical and on level ground, triangle ABC is similar to triangle DEF and their corresponding sides are therefore proportional. Thus the height of AC may be found as follows:

$$\frac{AC}{12} = \frac{20}{16}$$

$$AC = \frac{(12)(20)}{16} = 15$$

Practice problems. In each of the following problems, set up a proportion and then solve for the unknown quantity:

1. Referring to figure 13-1, if the shortest side of the larger triangle is 16 units long, rather than 10, how long is side c?

2. If a mast 8 ft high casts a shadow 10 ft long, how high is a mast that casts a shadow 40 ft long?

Answers:

1. $\frac{8}{16} = \frac{11}{c}$

$$8c = (11)(16)$$

$$c = \frac{(11)(16)}{8}$$

$$c = 22$$

2. $\frac{8}{10} = \frac{h}{40}$

$$\frac{(8)(40)}{10} = h$$

$$h = 32$$

Word Problems

A knowledge of proportions often provides a quick method of solving word problems. The following problem is a typical example of the types that lend themselves to solution by means of proportion.

If an automobile runs 36 mi on 2 gal of gas, how many miles will it run on 12 gal? Comparing miles to miles and gallons to gallons, we have

$$36:x = 2:12$$

Rewriting this in fraction form, the solution is as follows:

$$\frac{36}{x} = \frac{2}{12}$$

$$2x = 12(36)$$

$$x = 6(36)$$

$$= 216 \text{ mi}$$

Practice problems. In each of the following problems, first set up a proportion and then solve for the unknown quantity:

1. The ratio of the speed of one aircraft to that of another is 2 to 5. If the slower aircraft has a speed of 300 knots, what is the speed of the faster aircraft?

2. If 6 seamen can empty 2 cargo spaces in 1 day, how many spaces can 150 seamen empty in 1 day?

3. On a map having a scale of 1 in. to 50 mi, how many inches represent 540 mi?

Answers:

1. 750 kt 2. 50 3. 10.8 in.

VARIATION

When two quantities are interdependent, changes in the value of one may have a predictable effect on the value of the other. Variation is the name given to the study of the effects of changes among related quantities. The three types of variation which occur frequently in the study of scientific phenomena are DIRECT, INVERSE, and JOINT.

DIRECT VARIATION

An example of direct variation is found in the following statement: The perimeter (sum of the lengths of the sides) of a square increases if the length of a side increases. In everyday language, this statement might become: The longer the side, the bigger the square. In mathematical symbols, using p for perimeter and s for the length of the side, the relationship is stated as follows:

$$p = 4s$$

Since the number 4 is constant, any variations which occur are the results of changes in p and s. Any increase or decrease in the size of s results in a corresponding increase or decrease in the size of p. Thus p varies in the same way (increasing or decreasing) as s. This explains the terminology which is frequently used: p varies directly as s.

In general, if a quantity can be expressed in terms of a second quantity multiplied by a constant, it is said to VARY DIRECTLY AS the second quantity. For example if x and y are variables and k is a constant, x varies directly as y, if $x = ky$. Thus, as y increases x increases, and as y decreases, x decreases. There is a direct effect on x caused by any change in y.

The fact that x varies as y is sometimes indicated by $x \propto y$, or $x \sim y$. However, it is usually written in the form $x = ky$.

The relationship $x = ky$ is equivalent to $\frac{x}{y} = k$. If one quantity varies directly as a second quantity, the ratio of the first quantity to the second quantity is a constant. Thus, whatever the value of x, where it is divided by y, the result will always be the same value, k.

A quantity that varies directly as another quantity is also said to be DIRECTLY PROPORTIONAL to the second quantity. In $x = ky$, the coefficient of x is 1. The relationship $x = ky$ can be written in proportion form as

$$\frac{x}{k} = \frac{y}{1}$$

or

$$\frac{k}{x} = \frac{1}{y}$$

Notice that the variables, x and y, appear either in the numerators or in the denominators of the equal ratios. This implies that x and y are directly proportional. The constant, k, is the CONSTANT OF PROPORTIONALITY.

Practice problems. Write an equation showing the stated relationship, in each of the following problems:

1. The cost, C of a dozen wrenches varies directly as the price, p, of one wrench.

2. X is directly proportional to Y (use k as the constant of proportionality).

3. The circumference, C, of a circle varies directly as its diameter, d (use π as the constant of proportionality).

In the following problems, based on the formula $p = 4s$, find the appropriate word or symbol to fill the blank.

4. When s is doubled, p will be____________.

5. When s is halved, p will be____________.

6. ____________ is directly proportional to s.

Answers:

1. $C = 12p$
2. $X = kY$
3. $C = \pi d$
4. doubled
5. halved
6. p

Variation as the Power of a Quantity

Another form of direct variation occurs when a quantity varies as some power of another. For example, consider the formula

$$A = \pi r^2$$

Table 13-1 shows the values of r and the corresponding values of A.

Table 13-1.—Relation between values of radius and area in a circle.

When r = ----	1	2	3	4	5	7	9
Then A = ----	π	4π	9π	16π	25π	49π	81π

Notice how A changes as a result of a change in r. When r changes from 1 to 2, A changes from π to 4 times π or 2^2 times π. Likewise when r changes from 3 to 4, A changes not as r, but as the SQUARE of r. In general, one quantity varies as the power of another if it is equal to a constant times that quantity raised to the power. Thus, in an equation such as $x = ky^n$, x varies directly as the n^{th} power of y. As y increases, x increases but more rapidly than y, and as y decreases, x decreases, but again more rapidly.

Practice problems.

1. In the formula $V = e^3$, how does V vary?

2. In the formula $A = s^2$, if s is doubled how much is A increased?

3. In the formula $s = \frac{gt^2}{2}$, g is a constant. If t is halved, what is the resulting change in s?

Answers:

1. Directly as the cube of e.
2. It is multiplied by 4.
3. It is multiplied by $\frac{1}{4}$.

INVERSE VARIATION

A quantity VARIES INVERSELY as another quantity if the product of the two quantities is a constant. For example, if x and y are variables and k is a constant, the fact that x varies inversely as y is expressed by

$$xy = k$$

or

$$x = \frac{k}{y}$$

If values are substituted for x and y, we see that as one increases, the other must decrease, and vice versa. Otherwise, their product will not equal the same constant each time.

If a quantity varies inversely as a second quantity, it is INVERSELY PROPORTIONAL to the second quantity. In $xy = k$, the coefficient of k is 1. The equality $xy = k$ can be written in the form

$$\frac{x}{k} = \frac{1}{y}$$

or

$$\frac{k}{x} = \frac{y}{1}$$

Notice that when one of the variables, x or y, occurs in the numerator of a ratio, the other variable occurs in the denominator of the second ratio. This implies that x and y are inversely proportional.

Inverse variation may be illustrated by means of the formula for area of a rectangle. If A stands for area, L for length, and W for width, the expression for the area of a rectangle in terms of the length and width is

$$A = LW$$

Suppose that several rectangles, all having the same area but varying lengths and widths, are to be compared. Then $LW = A$ has the same form as $xy = k$, where A and k are constants. Thus L is inversely proportional to W, and W is inversely proportional to L.

If the constant area is 12 sq ft, this relationship becomes

$$LW = 12$$

If the length is 4 ft, the width is found as follows:

$$W = \frac{12}{L} = \frac{12}{4} = 3 \text{ ft}$$

If the length increases to 6 ft, the width decreases as follows:

$$W = \frac{12}{6} = 2 \text{ ft}$$

If a constant area is 12, the width of a rectangle decreases from 3 to 2 as the length increases from 4 to 6. When two inversely proportional quantities vary, one decreases as the other increases.

Another example of inverse variation is found in the study of electricity. The current flowing in an electrical circuit at a constant potential varies inversely as the resistance of the circuit. Suppose that the current, I, is 10 amperes when the resistance, R, is 11 ohms and it is desired to find the current when the resistance is 5 ohms.

Since I and R vary inversely, the equation for the relationship is $IR = k$, where k is the constant voltage. Therefore, $(10)(11) = k$. Also, when the resistance changes to 5 ohms, $(5)(I) = k$. Quantities equal to the same quantity are equal to each other, so we have the following equation:

$$5I = (10)(11)$$

$$I = \frac{110}{5} = 22$$

The current is 22 amperes when the resistance is 5 ohms. As the resistance decreases from 11 to 5 ohms, the current increases from 10 to 22 amperes.

One type of variation problem which tends to be confusing to the beginner involves rates of speed or rates of doing work. For example, if 7 men can complete a job in 20 days, how long will 50 men require to complete the same job? The strictly mechanical approach to this problem might result in the following false solution, relating men to men and days to days:

$$\frac{7 \text{ men}}{50 \text{ men}} = \frac{20 \text{ days}}{T}$$

However, a little thought brings out the fact that we are dealing with an INVERSE relationship rather than a direct one. In other words, the more men we have, the less time is required. Therefore, the correct solution requires that we use an inverse proportion; that is, we must invert one of the ratios as follows:

$$\frac{7}{50} = \frac{T}{20}$$

$$T = \frac{(7)(20)}{50} = 2\frac{4}{5} \text{ days}$$

Practice problems. In problems 1 and 2, express the given data as a proportion, using k as the constant of proportionality.

1. The rate, r, at which a vessel travels in going a certain distance varies inversely as the time, t.

2. The volume, V, of a gas varies inversely as the pressure, p.

3. A ship moving at a rate of 15 knots requires 10 hr to travel a certain distance. If the speed is increased to 25 knots, how long will the ship require to travel the same distance?

Answers:

1. $\frac{r}{k} = \frac{1}{t}$ 2. $\frac{V}{k} = \frac{1}{p}$ 3. 6 hr

JOINT VARIATION

A quantity VARIES JOINTLY as two or more quantities, if it equals a constant times their product. For example, if x, y, and z are variables and k is a constant, x varies jointly as y and z, if $x = kyz$. Note that this is similar to direct variation, except that there are two variable factors and the constant with which to contend in the one number; whereas in direct variation, we had only one variable and the constant. The equality, $x = kyz$, is equivalent to

$$\frac{x}{yz} = k$$

If a quantity varies jointly as two or more other quantities, the ratio of the first quantity to the product of the other quantities is a constant.

The formula for the area of a rectangle is an example of joint variation. If A is allowed to vary, rather than being constant as in the example used earlier in this chapter, then A varies jointly as L and W. When the formula is written for general use, it is not commonly expressed as $A = kLW$, although this is a mathematically correct form. Since the constant of proportionality in this case is 1, there is no practical need for expressing it.

Using the formula $A = LW$, we make the following observations: If $L = 5$ and $W = 3$, then $A = 3(5) = 15$. If $L = 5$ and $W = 4$, then $A = 4(5) = 20$, and so on. Changes in the area of a rectangle depend on changes in either the length or the width or both. The area varies jointly as the length and the width.

As a general example of joint variation, consider the expression $a \propto bc$. Written as an equation, this becomes $a = kbc$. If the value of a is known for particular values of b and c, we can find the new value of a corresponding to changes in the values of b and c. For example, suppose that a is 12 when b is 3 and c is 2. What is the value of a when b is 4 and c is 5? Rewriting the proportion,

$$\frac{a}{bc} = k$$

Thus

$$\frac{12}{(3)(2)} = k$$

Also,

$$\frac{a}{(4)(5)} = k$$

Since quantities equal to the same quantity are equal to each other, we can set up the following proportion:

$$\frac{a}{(4)(5)} = \frac{12}{(3)(2)}$$

$$a = 40$$

Practice problems. Using k as the constant of proportionality, write equations that express the following statements:

1. Z varies jointly as x and y.

2. S varies jointly as b times the square of r.

3. The length, W, of a radio wave varies jointly as the square root of the inductance, L, and the capacitance, C.

Answers:

1. $Z = kxy$

2. $S = kbr^2$

3. $W = k\sqrt{LC}$

COMBINED VARIATION

The different types of variation can be combined. This is frequently the case in applied problems. The equation

$$E = \frac{kw^2L}{p^2}$$

is an example of combined variation and is read, "E varies jointly as L and the square of W, and inversely as the square of p." Likewise,

$$V = \frac{krs}{t}$$

is read, "V varies jointly as r and s and inversely as t."